Algebraic Geometry I
Winter term 2021/2022 Florian Strunk

Exercise sheet 13

To be handed in until Wednesday, 2nd of February, 4pm (**only extra-points**)

Exercise 1 (4 Points) Let A be a ring and $I \subseteq A$ an ideal.

1. Show that $\mathrm{Ann}_A(A/I) = I$.
2. Let $\varphi: A \to B$ be a ring homomorphism. Show that there is an adjunction

$$(-) \otimes_A B\colon\ A\textbf{-Mod} \leftrightarrows B\textbf{-Mod} : \mathrm{res}_\varphi(-)$$

between the extension and the restriction of scalars.

Exercise 2 (4 Points) Let $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of schemes.

1. Show that the inverse image f^* preserves quasi-coherent modules.
2. Show that the direct image f_* preserves quasi-coherent modules, if X is a noetherian topological space.
(*Hint:* Restrict to the affine case in which this holds by Lemma 14.30.(2).)

Exercise 3 (4 Points) Let $f = (f, f^\sharp): (Z, \mathcal{O}_Z) \to (X, \mathcal{O}_X)$ be a morphism of schemes. Show that the property of f being a closed immersion can be checked "affine locally on the target", i.e. show that f is a closed immersion if there is an open affine covering $\{\mathrm{Spec}(A_i) = U_i \hookrightarrow X\}$ such that every induced $f^{-1}(U_i) \to U_i$ is a closed immersion.

Exercise 4 (4 Points) Consider a scheme S, a scheme $X \in \textbf{Sch}/S$ and the fibre product $X \times_S X$. The universal property induces a morphism

$$\Delta\colon\ X \to X \times_S X$$

of schemes. On topological spaces, this yields a subset

$$\Delta(X) \subseteq \{y \in X \times_S X \mid p_1(y) = p_2(y)\}$$

where p_1 and p_2 denote the two projections, respectively. Show that this inclusion need not be an equality.
(*Hint*: A candidate for such an X can be obtained by glueing two affine lines $\mathbb{A}^1$ along the open subsets $\mathbb{A}^1 \setminus \{0\}$.)