 Florian Strunk

Exercise sheet 12

To be handed in until Wednesday, 26th of January, 4pm

Exercise 1 (4 Points) Show the last part of Theorem 8.9, i.e. show that the counit

$$f^- f_* \to \mathrm{id}$$

of the adjunction $f^- : \mathbf{Sh}^{\mathcal{S}}(Y) \leftrightarrows \mathbf{Sh}^{\mathcal{S}}(X) : f_*$ is an isomorphism if $f: X \to Y$ is an open embedding or a closed embedding.

Exercise 2 (4 Points)

1. Give an example of ring homomorphism $\varphi: A \to B$ with an induced map of affine schemes $f: X \to Y$ where $X = \mathrm{Spec}(B)$ and $Y = \mathrm{Spec}(A)$ and an example of quasicoherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{F}'$ such that the direct image f_* does not commute with tensor products, i.e. such that the canonical map

$$f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} f_*(\mathcal{F}') \longrightarrow f_*(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{F}')$$

   is not an isomorphism.

2. Show that the canonical map from above is in fact an isomorphism if the ring homomorphism $\varphi: A \twoheadrightarrow A/I$ is surjective.

Exercise 3 (4 Points) Let $X := \mathrm{Spec}(\mathbb{Z}_{(p)})$ for a prime p. Then $|X| = \{\sigma, \eta\}$ as a topological space where σ is the closed point and η the generic point. Note that $\mathcal{O}_{X,\eta} \cong \mathbb{Q}$ is the quotient field of X.

1. Show that an $\mathcal{O}_X$-module $\mathcal{F}$ is given uniquely by the following data:
   (a) A $\mathbb{Z}_{(p)}$-module M,
   (b) a $\mathbb{Q}$-module N,
   (c) a $\mathbb{Q}$-module homomorphism $\nu: M \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \to N$

2. Show that an $\mathcal{O}_X$-module $\mathcal{F}$ is quasicoherent iff the corresponding map ν from above is an isomorphism.

(*Comment:* This gives more examples of non-quasicoherent $\mathcal{O}_X$-modules, e.g. if ν is zero.)

Exercise 4 (4 Points) Let $X = \mathrm{Spec}(A)$ be an affine scheme and

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

and exact sequence of $\mathcal{O}_X$-modules. Show that if two out of these three $\mathcal{O}_X$-modules are quasicoherent then so is the third.
(*Hint:* Use Theorem 14.29 in the case that $\mathcal{F}'$ and $\mathcal{F}''$ are quasicoherent.)
(*Comment:* This holds in fact by the same argument for a general scheme X.)