Algebraic Geometry I
Winter term 2021/2022 Florian Strunk

Exercise sheet 11

To be handed in until Wednesday, 19th of January, 4pm

Exercise 1 (4 Points) Consider a prime $p \in \mathbb{Z}$ and $n \geq 1$ an integer. Let $\mathbb{F}_{p^n}$ denote the field with exactly p^n elements.

1. Describe the topological spaces and the structure sheaves of the schemes $\mathrm{Spec}(\mathbb{F}_{p^n})$ and $\mathrm{Spec}(\mathbb{Z}/p^n)$ respectively.
2. How many points of $\mathbb{A}^1_{\mathbb{F}_3}$ have residue field $\mathbb{F}_{27}$ and what residue fields do appear?

Exercise 2 (4 Points)

1. Let $f: X \to Y$ be a morphism of schemes over a base scheme S, and let $\Gamma_f: X \to X \times_S Y$ be the *graph morphism*, defined by the universal property of the fiber product and the two morphisms $f: X \to Y$, $\mathrm{id}: X \to X$. Show that the diagram
$$\begin{tikzcd} X \arrow[r, "\Gamma_f"] \arrow[d, "f"] & X \times_S Y \arrow[d, "f \times \mathrm{id}"] \\ Y \arrow[r, "\Delta"] & Y \times_S Y \end{tikzcd}$$
is a fiber product.
2. Let I and J be ideals of a ring A. Show that the canonial diagram
$$\begin{tikzcd} \mathrm{Spec}(A/(I+J)) \arrow[r, "h"] \arrow[d, "k"] & \mathrm{Spec}(A/I) \arrow[d, "f"] \\ \mathrm{Spec}(A/J) \arrow[r, "g"] & \mathrm{Spec}(A) \end{tikzcd}$$
is a fiber product.

Exercise 3 (4 Points) Let k be a field. Show that the diagram
$$\begin{tikzcd} k[x,y]/(xy) \arrow[r] \arrow[d] & k[y] \arrow[d, "y \mapsto 0"] \\ k[x] \arrow[r, "x \mapsto 0"] & k \end{tikzcd}$$
is a fiber product in the category of Rings where the initial maps are the quotient maps when we write $k[y] \cong k[x,y]/(x)$ and $k[x] \cong k[x,y]/(y)$.

Exercise 4 (4 Points) Let A be a ring. Show the following assertions.

1. For an element $s \in A$, we have $A[\frac{1}{s}] \cong \mathrm{colim}(A \xrightarrow{\cdot s} A \xrightarrow{\cdot s} A \xrightarrow{\cdot s} \cdots)$ in the category **Ab**.
2. For $\mathfrak{p} \in \mathrm{Spec}(A)$, we have $A_{\mathfrak{p}} \cong \mathrm{colim}_{D(s) \ni \mathfrak{p}} A[\frac{1}{s}]$ where the structure maps are a unique arrow $D(s) \to D(t)$ if and only if $D(t) \subseteq D(s)$. (Check also that this is well definied, i.e. the ring $A[\frac{1}{s}]$ depends (up to canonical isomorphism) only on $D(s)$ and not only on the element s.)