Algebraic Geometry I
Winter term 2021/2022

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Exercise sheet 10

To be handed in until Wednesday, 12th of January, 4pm

Exercise 1 (4 Points)

1. Show Lemma 11.23. which asserts that the topological space X which underlies a scheme is a T_0 space, i.e. for two distinct points $x \neq y$ of X there is an open subset $U \subseteq X$ containing exactly one of the two points.

2. Show that for a topological space X we have the equivalence

$$X \text{ is a noetherian space} \quad \Longleftrightarrow \quad \text{Every open subset } U \subseteq X \text{ is quasicompact.}$$

3. Find an example of an open subscheme of an affine (hence quasicompact) scheme which is not quasicompact.

4. Show that every quasicompact scheme has a closed point (i.e. $x \in X$ with $\overline{\{x\}} = x$).

Exercise 2 (2 Points) Let p be a prime number, $\mathbb{F}_p := \mathbb{Z}/p$ and X a scheme. Show the following equivalences.

$$\begin{aligned} \mathcal{O}_X(X) \text{ has characteristic } p \quad &\Longleftrightarrow \quad \mathcal{O}_X(U) \text{ has characteristic } p \text{ for every } U \subseteq X \text{ open} \\ &\Longleftrightarrow \quad \text{The canonical morphism } X \to \operatorname{Spec}(\mathbb{Z}) \text{ factorizes} \\ &\qquad\quad\; \text{over the canonical morphism } \operatorname{Spec}(\mathbb{F}_p) \hookrightarrow \operatorname{Spec}(\mathbb{Z}) \\ &\qquad\quad\; \text{induced by the quotient map } \mathbb{Z} \twoheadrightarrow \mathbb{Z}/p. \end{aligned}$$

Exercise 3 (2 Points) Show Lemma 12.6. which asserts that in a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d] \\ S'' \arrow[r] & S' \arrow[r] & S \end{tikzcd}$$

where the right-hand side square is a fiber product (as indicated by the symbol "$\lrcorner$"), the outer rectangle is a fiber product if and only if the left-hand side square is a fiber product.

Exercise 4 (4 Points) Consider the morphism

$$f\colon \quad X := \operatorname{Spec}(\mathbb{Q}[x,y]/(x-y^2)) \to S := \operatorname{Spec}(\mathbb{Q}[t])$$

of schemes induced by $t \mapsto x$ (draw a picture!).

1. Calculate the fibers X_s of f at all points $s := (t-a) \in S$ for $a \in \mathbb{Q}$.
2. Calculate the fiber X_η of f at the generic point $\eta \in S$.

Exercise 5 (4 Points) Fix positive integers α, β and γ and consider the *Fermat scheme*

$$S := \operatorname{Spec}(\mathbb{Z}[x,y,z]/(x^\alpha + y^\beta - z^\gamma))$$

with open subscheme $U := S \setminus \mathcal{V}((x,y,z)+(x^\alpha + y^\beta - z^\gamma))$. Show that the set of morphisms

$$\operatorname{Hom}_{\mathbf{Sch}}(\operatorname{Spec}(\mathbb{Z}), U)$$

is in bijection with the integer solutions $(a,b,c) \in \mathbb{Z}^3$ to $x^\alpha + y^\beta = z^\gamma$ with $ggT(a,b,c) = 1$.