Algebraic Geometry I
Winter term 2021/2022 Florian Strunk

Exercise sheet 09

To be handed in until Wednesday, 22th of December, 4pm

Exercise 1 (4 Points)

1. Show that a morphism $(f,f^\sharp)\colon(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ of schemes is an isomorphism (i.e. there exists an inverse $(g,g^\sharp)\colon(Y,\mathcal{O}_Y)\to(X,\mathcal{O}_X)$ such that both compositions are the identity) if and only if f is an isomorphism (i.e. a homeomorphism) and $f^\sharp$ is an isomorphism (of sheaves).

2. Let $\{V_i\hookrightarrow Y\}$ be an open covering. Show that a morphism $(f,f^\sharp)\colon(X,\mathcal{O}_X)\to(Y,\mathcal{O}_Y)$ of schemes is an isomorphism if and only if for all indices i the induced morphism $(f^{-1}(V_i),\mathcal{O}_{X|f^{-1}(V_i)})\to(V_i,\mathcal{O}_{Y|V_i})$ is an isomorphism.

Exercise 2 (4 Points) Consider the scheme $X:=\mathrm{Spec}(\mathbb{Q}[x,y]/(xy))$ and the scheme $Y:=\mathrm{Spec}(\mathbb{Q}[x,y]/(x^2+y^2))$.

1. Describe $\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(\mathbb{Q}),X)$ and $\mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}(\mathbb{Q}),Y)$ as subsets of $\mathbb{Q}^2$ via Theorem 10.19 and Lemma 3.3.

2. Deduce that $X\ncong Y$ as schemes.

Consider the Schemes $X:=\mathrm{Spec}(\mathbb{Q}(i)[x,y]/(xy))$ and $Y:=\mathrm{Spec}(\mathbb{Q}(i)[x,y]/(x^2+y^2))$.

3. Show that $X\cong Y$ as schemes.

Exercise 3 (4 Points)

1. Let A be a ring and $s\in A$ an element. Show that the homeomorphism of topological spces $t\colon\mathrm{Spec}(A[\frac{1}{s}])\xrightarrow{\cong}D(s)$ with $D(s)\subseteq\mathrm{Spec}(A)$ of Lemma 4.10.(2) induces an isomorphism
$$(\mathrm{Spec}(A[\tfrac{1}{s}]),\mathcal{O}_{\mathrm{Spec}(A[\frac{1}{s}])})\cong(D(s),\mathcal{O}_{\mathrm{Spec}(A)|D(s)})$$
of schemes, as claimed in the proof of Lemma 11.8. (Note that here $D(s)$ is considered as in Definition 4.3.)

2. Let $(X,\mathcal{O}_X)$ be a scheme, $f\in\mathcal{O}_X(X)$ and $\mathrm{Spec}(A)\cong U\hookrightarrow X$ an affine open subscheme. Let moreover s be the image of f under the restriction morphism $\mathcal{O}_X(X)\to\mathcal{O}_X(U)\cong\mathcal{O}_U(U)\cong\mathrm{Spec}(A)$. Show that
$$D(f)\cap\mathrm{Spec}(A)=D(s)$$
where the $D(f)$ of the left-hand side is from Lemma 10.17, the intersection takes place in X, and the $D(s)$ on the right-hand side is from Definition 4.3.

Exercise 4 (4 Points) Show that a scheme $(X, \mathcal{O}_X)$ is affine if and only if there exist finitely many elements $s_1, \ldots, s_n \in A := \mathcal{O}_X(X)$ with $D(s_i) \subseteq X$ affine and which generate the unit ideal of A.

Hints:

- Show that $X = \cup_i D(s_i)$.
- Show with exercise 1 and Thm. 10.19. and glueing of morphisms (Corollary 11.15) that it suffices to prove $D(s_i) \cong \mathrm{Spec}(A[\frac{1}{s_i}])$.
- Show the latter isomorphism (this is not trivial: certain elements, „local sections", have to be glued to an element in $A = \mathcal{O}_X(X)$, a „global section").