Algebraic Geometry I
Winter term 2021/2022 Florian Strunk

Exercise sheet 08

To be handed in until Wednesday, 15th of December, 4pm

Exercise 1 (2 Points) Let $\varphi: A \to B$ be a ring-map between two local rings $(A, \mathfrak{m}_A)$ and $(B, \mathfrak{m}_B)$. Show Remark 10.12, i.e. show the equivalence

$$\varphi(\mathfrak{m}_A) \subseteq \mathfrak{m}_B \quad \Longleftrightarrow \quad \varphi^{-1}(\mathfrak{m}_B) = \mathfrak{m}_A.$$

Exercise 2 (4 Points) Find an open subscheme of $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x])$ which is not affine (with a proof, of course). Does such an example exist also for $\mathbb{A}^1_k = \mathrm{Spec}(k[x])$ where k is a field?

Exercise 3 (6 Points) Let $\varphi: A \to B$ a ring homomorphism and consider the induced morphism $(\tilde{\varphi}, \tilde{\varphi}^\sharp): \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of ringed spaces. Show that

$$\varphi \text{ is injective} \quad \Longleftrightarrow \quad \tilde{\varphi}^\sharp \text{ is stalkwise injective}$$

and

$$\varphi \text{ is surjective} \quad \Longleftrightarrow \quad \begin{array}{l} \tilde{\varphi} \text{ is a closed embedding and} \\ \tilde{\varphi}^\sharp \text{ is stalkwise surjective.} \end{array}$$

Exercise 4 (4 Points) Let k be a field and $U_1 := \mathrm{Spec}(k[x])$ and $U_2 := \mathrm{Spec}(k[y])$ both being $\mathbb{A}^1_k$. Consider the open subschemes $U_{12} := \mathrm{Spec}(k[x, x^{-1}])$ and $U_{21} := \mathrm{Spec}(k[y, y^{-1}])$ both being $\mathbb{A}^1_k \setminus \{0\}$. Let X be the scheme obtained by glueing U_1 and U_2 along

$$\begin{array}{cccc} \phi_{12}\colon & U_{12} & \xrightarrow{\cong} & U_{21} \\ & y & \mapsto & x \end{array}$$

Show that X is not an affine scheme and try to draw an intuitive picture.