Algebraic Geometry I
Winter term 2021/2022 Florian Strunk

Exercise sheet 04

To be handed in until Wednesday, 17th of November, 4pm

Exercise 1 (4 Points) Show that $\mathbb{C}^2$ is not homeomorphic to $\mathbb{C}^1 \times \mathbb{C}^1$ in the classical $\mathbb{C}$-Zariski Topology.

(What about the same question for the usual topology on $\mathbb{C}^2$ and $\mathbb{C}^1$? Recall that the open sets in the product topology on $X \times Y$ are the unions of open sets of the form $U \times V$ for $U \subseteq X$ and $V \subseteq Y$ open.)

Exercise 2 (4 Points) Decide (this means always: with a proof) whether the canonical morphism $\coprod_{\mathbb{N}} \operatorname{Spec}(\mathbb{F}_2) \to \operatorname{Spec}(\prod_{\mathbb{N}} \mathbb{F}_2)$ is a homeomorphism in the Zariski topology.

Exercise 3 (4 Points) Let k be a field.

1. Show that $\operatorname{Spec}(k[x_1, \ldots, x_n])$ is infinite if $n \geq 1$.
2. Let A be a k-algebra of finite type with $\operatorname{Spec}(A)$ finite. Conclude that A is finite over k.

Exercise 4 (4 Points) Let X be a compact Hausdorff space and set

$$\mathcal{C}(X, \mathbb{R}) := \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}.$$

This is a ring by adding and multiplying value-wise. Consider the map

$$\begin{array}{rccl} \mu\colon & X & \to & \operatorname{mSpec}(\mathcal{C}(X,\mathbb{R})) \\ & x & \mapsto & \mathfrak{m}_x := \{f \in \mathcal{C}(X,\mathbb{R}) \mid f(x) = 0\}. \end{array}$$

and show that this is a homeomorphism where $\operatorname{mSpec}(\mathcal{C}(X,\mathbb{R}))$ carries the subspace topology of the Zariski topology on $\operatorname{Spec}(\mathcal{C}(X,\mathbb{R}))$. (Remark: As a non-trivial input, this needs Urysohn's lemma from topology: the continuous functions separate the points of X. This lemma can be used without proof or reference.)