Algebraic Geometry I
Winter term 2021/2022 Florian Strunk

Exercise sheet 02

To be handed in until Wednesday, 3rd of November, 4pm

Exercise 1 (4 Points)

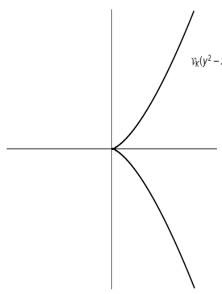
1. *(The image of an algebraic set under an algebraic map may not be an algebraic set)*
   Find for $R = K = \mathbb{C}$ an example of an ideal $I \subseteq R[x_1,\dots,x_n]$ and an R-algebraic map
   $$\bar{F}: \mathcal{V}_K(I) \to K^m$$
   such that the image $\bar{F}(\mathcal{V}_K(I))$ is not of the form $\mathcal{V}_K(J)$ for some ideal $J \subseteq R[y_1,\dots,y_m]$.

2. *(The preimage of an algebraic set under an algebraic map is an algebraic set)*
   Let R be a ring and $R \to K$ an algebra. Show that for any ideal $J \subseteq R[y_1,\dots,y_m]$ and any R-algebraic map
   $$\bar{F}: K^n \to \mathcal{V}_K(J)$$
   the preimage $\bar{F}^{-1}(\mathcal{V}_K(J))$ is of the form $\mathcal{V}_K(I)$ for some ideal $I \subseteq R[x_1,\dots,x_n]$.

Exercise 2 (4 Points) *(A bijective algebraic map that is not an algebraic isomorphism)*
Let $R = K = \mathbb{R}$ and $W := \mathcal{V}_K(y^2 - x^3) \subseteq K^2$. Show that the R-algebraic map

$$\begin{array}{cccc} \bar{F}: & K^1 & \to & W \\ & t & \mapsto & (t^2, t^3) \end{array}$$

is bijective but not an algebraic isomorphism.



(This does not show, even though it is true, that $K^1 \ncong W$.)

Exercise 3 (4 Points) Show that the twisted cubic $W := \mathcal{V}_K(x^2 - y, x^3 - z) \subseteq K^3$ is R-algebraically isomorphic to K^1.

Exercise 4 (4 Points) Let k be a field and consider the k-subalgebra

$$A := \{f \in k[t] \mid f(0) = f(1)\}$$

of $k[t]$. Find a representation $A \cong k[x,y]/I$ for some ideal I.
(This shows in particular that A is a k-Algebra of finite type.)