Algebraic Geometry I
Winter term 2021/2022 Florian Strunk

Exercise sheet 01

To be handed in until Wednesday, 27th of October, 4pm

Exercise 1 (4 Points) Show that being reduced is a local property for rings, i.e. show that for a ring A the equivalence

$$A \text{ is reduced } \Leftrightarrow \text{For all } \mathfrak{p} \in \operatorname{Spec}(A) \text{ the ring } A_{\mathfrak{p}} \text{ is reduced}$$

holds.

Exercise 2 (4 Points) Show that $\operatorname{Spec}(A \times B) = \operatorname{Spec}(A) \coprod \operatorname{Spec}(B)$ and deduce that being irreducible (i.e. having exactly one minimal prime ideal) is not a local property for rings.

Exercise 3 (4 Points) Let A be a noetherian ring and $\mathfrak{p}' \subsetneq \mathfrak{p} \subsetneq \mathfrak{p}''$ a chain of prime ideals. Show that there are infinitely many prime ideals $\mathfrak{p}$ with this property.
(Hint: Use prime avoidance and Krull's height theorem.)

Exercise 4 (4 Points) We have shown that the maximal ideals of $k[x_1,\dots,x_n]$ are exactly of the form $I_{\underline{a}} = (x_1 - a_1,\dots,x_n - a_n)$ if $k = \bar{k}$ is an algebraically closed field. Every polynomial $x_j - a_j \in k[x_j] \subseteq k[x_1,\dots,x_n]$ is an irreducible element.

1. Show that for $k = \mathbb{Q}$, an ideal in $k[x_1,\dots,x_n]$ generated by n-many irreducible polynomials
$$f_j \in k[x_j] \subseteq k[x_1,\dots,x_n].$$
may not be maximal.

2. Show that, for an arbitrary field k, the maximal ideals in $k[x_1,\dots,x_n]$ are generated by n-many irreducible polynomials $f_j \in k[x_1,\dots,x_n]$.
(Hint: Use induction on n.)

   (Discussion for the exercise sessions: Are the maximal ideals in $k[x_1,\dots,x_n]$ exactly the ideals generated by n-many certain irreducible polynomials f_j? What about $f_j \in k[x_1,\dots,x_j]$ for example?)