Exercise sheet 19

Algebraic Geometry II
Summer term 2018

Exercise 1

An affine morphism $f : X \to Y$ is called *finite* if for all affine open $U \subset Y$, the induced map

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$$

exhibits $\mathcal{O}_X(f^{-1}U)$ as finite $\mathcal{O}_Y(U)$-module.

1. Show that f is finite if and only if there is a cover $Y = \bigcup_i U_i$ by open affine schemes such that $\mathcal{O}_X(f^{-1}U_i)$ is a finite $\mathcal{O}_Y(U_i)$-module.
2. Give an example of a finite morphism which is neither injective nor surjective on underlying topological spaces.
3. Show that finiteness is stable under composition and base change.

Exercise 2

Calculate the dimension of

$$X := \operatorname{Proj}(k[x, y, z, w]/(xz - y^2, yz - xw, z^2 - yw))$$

where k is a field and polynomial rings are equipped with their usual gradings.

Exercise 3

Let $f : X \to Y$ be a map of schemes. For the following list of properties of f, decide with proof resp. counterexample which implications hold:

1. f is finite.
2. f is of finite type.
3. f is affine.
4. f is proper.
5. f is quasi-compact.
6. f is integral (see Ex. 4).
7. f is separated.
8. f is integral and locally of finite type.
9. f is a closed immersion.

Exercise 4

An affine morphism $f : X \to Y$ is called *integral* if for all affine open $U \subset Y$, the induced map

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$$

is an integral map of rings. Show that integral morphisms are universally closed.

Hint: You may use that injective integral ring maps $A \to B$ induce surjections $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.