Exercise sheet 18

Algebraic Geometry II
Summer term 2018

Exercise 1

Let L/K be a finite field extension and $x \in L$. Denote by $\mathrm{Norm}_{L/K}(x)$ the determinant of the K-vectorspace-homomorphism

$$L \to L,\ \ell \mapsto x \cdot \ell.$$

1. Show that $\mathrm{Norm}_{L/K} : L^\times \to K^\times$ is a group homomorphism.
2. Let $A \subset B$ be an integral ring extension with quotient fields $K = \mathrm{Frac}(A)$, $L = \mathrm{Frac}(B)$. Show that for any $b \in B$, the norm $\mathrm{Norm}_{L/K}(b)$ is integral over A.
3. In the setup of 2), assume further that $A = k[T]$ for a field k. Show that $\mathrm{Norm}_{L/K}(b) \in A$ for all $b \in B$.

Exercise 2

A morphism $f : X \to Y$ is called *affine* if preimages of affine opens of Y are again affine. Show that f is affine if and only if there is a cover $Y = \bigcup_i U_i$ by open affine schemes such that $f^{-1}U_i$ is affine.

Exercise 3

An affine morphism $f : X \to Y$ is called *finite* if for all affine open $U \subset Y$, the induced map

$$\mathcal{O}_Y(U) \to \mathcal{O}_X(f^{-1}U)$$

exhibits $\mathcal{O}_X(f^{-1}U)$ as finite $\mathcal{O}_Y(U)$-module.

1. Show that f is finite if and only if there is a cover $Y = \bigcup_i U_i$ by open affine schemes such that $\mathcal{O}_X(f^{-1}U_i)$ is a finite $\mathcal{O}_Y(U_i)$-module.
2. Give an example of a finite morphism which is neither injective nor surjective on underlying topological spaces.
3. Show that finiteness is stable under composition and base change.

Exercise 4

Calculate the dimension of

$$X := \mathrm{Proj}(k[x, y, z, w]/(xz - y^2, yz - xw, z^2 - yw))$$

where k is a field and polynomial rings are equipped with their usual gradings.