Exercise sheet 17

Algebraic Geometry II
Summer term 2018

Exercise 1

Let k be a field with algebraic closure $\bar{k}$ and let $K/\bar{k}$ be a field extension. For $X \to \operatorname{Spec}(k)$ locally of finite type, show that the following are equivalent:

1. $X = \emptyset$.
2. There is no morphism $\operatorname{Spec}(\bar{k}) \to X$.
3. There is no morphism $\operatorname{Spec}(K) \to X$.

Which implications become wrong if $X \to \operatorname{Spec}(k)$ is not necessarily locally of finite type?

Exercise 2

Let $n, q \in \mathbb{N}$ be positive integers and suppose we are given polynomials

$$f_1, \ldots, f_q \in \mathbb{Z}[X_1, \ldots, X_n].$$

Show that the following are equivalent:

1. The system $f_1(x) = \ldots = f_n(x) = 0$ has a solution $x \in \mathbb{C}^n$.
2. The system $f_1(x) = \ldots = f_n(x) = 0$ has a solution $x \in \bar{\mathbb{F}}_p^n$ for all but finitely many primes p.
3. The system $f_1(x) = \ldots = f_n(x) = 0$ has a solution $x \in \bar{\mathbb{F}}_p^n$ for infinitely many primes p.

Hint: Use Chevalley's theorem.

Exercise 3

Show that the property of being locally of finite type is stable under composition and base change.

Exercise 4

Let X, Y be schemes over a locally noetherian base S and suppose that the structure morphism $Y \to S$ is locally of finite type. Given a point $x \in X$ and an S-morphism

$$f_x : \operatorname{Spec}(\mathcal{O}_{X,x}) \to Y$$

show that there is an open neighbourhood $U \subset X$ of x and an S-morphism $f : U \to Y$ such that f_x factors as

$$\operatorname{Spec}(\mathcal{O}_{X,x}) \xrightarrow{i_x} U \xrightarrow{f} Y$$

where i_x is induced by the restriction $\mathcal{O}_X(U) \to \mathcal{O}_{X,x}$.