Exercise sheet 13

Algebraic Geometry II
Summer term 2018

Exercise 1

Let $f : X \to Y$ be a morphism of schemes over a base scheme S, and let $\Gamma_f : X \to X \times_S Y$ be the *graph morphism*, defined by the universal property of the fibered product and the two morphisms $f : X \to Y$, $\mathrm{id} : X \to X$. Show that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\Gamma_f} & X \times_S Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f \times \mathrm{id}} \\ Y & \xrightarrow{\Delta} & Y \times_S Y \end{array}$$

is cartesian.

Exercise 2

Show that sections of separated morphisms are closed immersions, i.e. show that given two morphisms of schemes $f : V \to W$ and $g : W \to V$ with f separated and $f \circ g = \mathrm{id}$, the morphism g is a closed immersion.
Hint: Use Exercise 1.

Exercise 3

Show that the composition and the base change of separated morphisms are again separated.

Exercise 4

Let A be a ring and $\mathbb{P}^n_A$ be projective n-space, defined as glueing $(n+1)$ copies of $\mathbb{A}^n_A$. Show that $\mathbb{P}^n_A$ is a separated scheme and show that it is an integral scheme iff A is a domain.

Exercise 5

Let A be a graded domain. Decide if the following statements are true or false.

1. Any unit is homogeneous.
2. Let $I \subset A$ be a homogeneous ideal. Then $\sqrt{I}$ is also a homogeneous ideal.
3. If the sum of two ideals is homogeneous, at least one of them is also homogeneous.
4. Let A be a field. Then its grading is trivial.
5. Every ideal I of A contains at least one non-zero homogeneous element.