Exercise sheet 7

Algebraic Geometry I
Winter term 2017/2018

Exercise 1

Let $(X, \mathcal{O}_X)$ be an arbitrary ringed space and $f \in \mathcal{O}_X(X)$ a global section. Show that the set

$$Y := \{x \in X \mid f_x \text{ is invertible}\}$$

is an open subset $Y \subseteq X$. Deduce the equivalence:

$$f \text{ is invertible } \Leftrightarrow \forall x \in X : f_x \text{ is invertible}$$

Exercise 2

Let X be a topological space and $\mathcal{F} \colon \mathrm{Ouv}(X)^{op} \to \mathbf{Ab}$ a presheaf of abelian groups on X. For a morphism $f \colon \mathcal{F} \to \mathcal{G}$ of presheaves of abelian groups on X, we define the *Kernel presheaf* $\mathrm{Ker}(f)$ by

$$U \mapsto \mathrm{Ker}(f(U))$$

and the *Image presheaf* $\mathrm{Im}(f)$ by

$$U \mapsto \mathrm{Im}(f(U))$$

both with the obvious restriction maps. Show that $\mathrm{Ker}(f)$ is a sheaf (also called the *Kernel sheaf*) if $\mathcal{F}$ and $\mathcal{G}$ are sheaves. Show that the same is not true for $\mathrm{Im}(f)$. The sheafification $a(\mathrm{Im}(f))$ is called the *Image sheaf* and by abuse of notation denoted by the same symbol. We say that a sequence

$$\mathcal{F}' \xrightarrow{f'} \mathcal{F} \xrightarrow{f} \mathcal{F}''$$

of presheaves (respectively sheaves) of abelian groups on X is *exact*, if $\mathrm{Im}(f') = \mathrm{Ker}(f)$ (where $\mathrm{Im}(f')$ denotes the image sheaf in the sheaf case). Suppose we have given an exact sequence

$$0 \to \mathcal{F}' \xrightarrow{f'} \mathcal{F} \xrightarrow{f} \mathcal{F}'' \to 0$$

of presheaves (respectively of sheaves) of abelian groups on X. Decide whether the induced sequence

$$0 \to \mathcal{F}'(X) \xrightarrow{f'(X)} \mathcal{F}(X) \xrightarrow{f(X)} \mathcal{F}''(X) \to 0$$

of abelian groups is exact or just partly exact.

Exercise 3

Let $f \colon A \hookrightarrow B$ be an an injective morphism of rings such that B is a finitely generated A-module (i.e., f is of finite type and integral). Show that the induced morphism $\mathrm{Spec}(f)\colon \mathrm{Spec}(B) \twoheadrightarrow \mathrm{Spec}(A)$ is surjective.

Exercise 4

Recall all the appering terms in the following statement, called the *Noether Normalization* (cf. Theorem A.26 of the Skript), and write down a complete proof:
Let k be a field, $A := k[X_1, \ldots, X_n]$ and $B := A/I$ for an ideal $I \neq A$. Then there exists algebraically independent elements $t_1, \ldots, t_d \in A$ such that $k[t_1, \ldots, t_d] \subseteq B$ is integral and injective.